INVARIANT SOLUTIONS OF EQUATIONS OF NONSTATIONARY MOTION OF GAS IN SPACE

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The system of equations which governs the nonstationary three-dimensional motion of polytropic gas is given by

$$D\mathbf{u} + (1/\rho)\nabla p = 0,\ D\rho + \rho \operatorname{div} \mathbf{u} = 0,\ Dp + \gamma p \operatorname{div} \mathbf{u} = 0, \tag{1}$$

where **u** is the velocity vector with the components u, v, w; p is the pressure; ρ is the density; γ is the adiabatic index; $D = \partial/\partial t + \mathbf{u}\cdot\nabla$ is the total-derivative operator.

The basic Lie transformation group was found in [1] and is consistent with such system. The basic Lie algebra operators for this group are as follows:

$$X_1 = \partial/\partial t,\ X_2 = \partial/\partial x,\ X_3 = \partial/\partial y,\ X_4 = \partial/\partial z,$$
$$X_5 = t\partial/\partial t + x\partial/\partial x + y\partial/\partial y + z\partial/\partial z,$$
$$X_6 = t\partial/\partial x + \partial/\partial u,\ X_7 = t\partial/\partial y + \partial/\partial v,$$
$$X_8 = t\partial/\partial z + \partial/\partial w,\ X_9 = t\partial/\partial t - u\partial/\partial u - v\partial/\partial v -$$
$$- w\partial/\partial w + 2\rho\partial/\partial\rho,\ X_{10} = z\partial/\partial y - y\partial/\partial z + w\partial/\partial v -$$
$$- v\partial/\partial w,\ X_{11} = x\partial/\partial z - z\partial/\partial x + u\partial/\partial w - w\partial/\partial u,$$
$$X_{12} = y\partial/\partial x - x\partial/\partial y + v\partial/\partial u - u\partial/\partial v,$$
$$X_{13} = p\partial/\partial p + \rho\partial/\partial\rho.$$

The group is said to be G_{13} and its Lie algebra L_{13}. The optimal subgroup systems of the group G_{13} were obtained in [2]. In Table 1 the basic operators are shown for the optimal systems of three-parameter subgroups of this group; however, the basic operators for these subalgebras in which the transfer operators in the spatial variables appear as generators are not shown since such subalgebras will not be considered by us (α, β, δ, ε, μ, ν are arbitrary constants). For the adiabatic index γ = 5/3 the base group is augmented and one adds to the operators

$$X_{14} = t^2\partial/\partial t + tx\partial/\partial x + ty\partial/\partial y + tz\partial/\partial z + (x - tu)\partial/\partial u +$$
$$+ (y - tv)\partial/\partial v + (z - tw)\partial/\partial w - 5tp\partial/\partial p - 3t\rho\partial/\partial\rho.$$

The augmented group is denoted by G_{14} and the corresponding Lie algebra by L_{14}.

The algebra L_{14} is not solvable. It possesses a nonvanishing radical which can be expressed by means of the operators X_2, X_3, X_4, X_6, X_7, X_8, X_{13}. Since the group G_{14} is unsolvable one cannot apply the algorithm described in [3] for finding optimal subgroups. Nevertheless, by using the optimal systems of subgroups of the group G_{13} enumerated in [2] one is able to find the optimal systems of the subgroups of the group G_{14}. The following proposition is now employed.

Proposition. Any subalgebra L_m of dimension m ≥ 2 of the algebra L_{14} is either a subalgebra of the algebra L_{13} or it contains a subalgebra L_{m-1} of dimension m − 1 which is a subalgebra of the algebra L_{13}.

Indeed, let us consider the basic operators of the algebra L_m: $Y_\alpha = \alpha_\alpha^i X_i$ α = 1, 2, ..., m (summation with respect to i is from 1 to 14). If α_α^{14} vanish for any α, then $L_m \subset L_{13}$. For some β let $\alpha\beta^{14} \neq 0$. It can be assumed without loss of generality that $\alpha_m^{14} = 1$. One now replaces the first m − 1 operators by the operators $Y'_\beta = Y_\beta - \alpha_\beta^{14} Y_m$ (β = 1, 2, ..., m − 1). The operators $Y'_1, Y'_2, \ldots, Y'_{m-1}$ then form a subalgebra of dimension m − 1 for the algebra L_{13}.

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TABLE 1.

X_1	X_5	X_9			
	$X_2 \pm X_{10}$				$X_5 + \alpha X_9$
	$\alpha X_5 + X_{10}$			$X_4 \pm X_7$	$X_5 - X_9$
	$\alpha X_9 + X_{10}$	$X_2 + X_9$		$X_2 \pm X_9$	$\alpha X_9 + X_{10}$
		$X_5 + \beta X_9$		$\alpha X_5 + X_9$	$\beta X_5 + X_{10}$
X_6	$X_5 - X_9$	$X_2 + \alpha X_4 \pm X_7$		$X_3 \pm X_9$	$X_7 + \alpha X_8$
		$\alpha X_2 + X_3 + \beta X_4 \pm X_7$		X_5	X_9
	$X_2 + \alpha X_3 + \beta X_4 \pm X_7$	$\mu X_2 + \delta X_3 + \varepsilon X_4 + X_8$	X_{10}	$X_1 + X_6$	$2X_5 - X_9$
	$X_3 + \alpha X_4 + X_7$	$\beta X_2 + \nu X_3 + \delta X_4 + X_8$		X_5	X_9
$X_3 \pm X_6$	$\alpha X_2 + \varepsilon X_3 + \mu X_4 + X_7$			X_7	X_8
X_6	X_7	X_8		X_{11}	X_{12}
		X_9			

Consequently, by taking the complement of each subalgebra of dimension m − 1 of the optimal system of the algebra L_{13} to the subalgebra of dimension m with the basic operators $Y'_1, Y'_2, \ldots, Y'_{m-1}, Y_m$ ($a_m^{14} = 1$), by simplifying the obtained operators by the inner-automorphism transformations of the group G_{14}, and finally by eliminating similar subalgebras from the set of the obtained subalgebras and of the subalgebras of dimension m of the algebra L_{13} one obtains an optimal system of m-parameter subgroups (not in [2]) of the group G_{14}.

Thus, optimal systems have been obtained of the three-parameter subgroups of the group G_{14}. In Table 2 the basis operators of these subgroups are shown; the basis operators for the subgroups similar to the subgroups of the groups G_{13} are not shown since they were enumerated in [2], as already mentioned; the subalgebras in which the transfer operators in the spatial variables appear as generators are not shown either since the invariant solutions obtained on the corresponding subgroups reduce to the solving of the equations of the two-dimensional gasdynamics analyzed in [3, 4]. As in [1, 3, 4], the operator X_{13}, which is the center of the algebra L_{14}, is not considered (α, β, δ are arbitrary constants).

We now proceed to find some invariant solutions of the system (1). First, the case of an arbitrary adiabatic index is considered. The solutions are obtained on the three-parameter subgroups of the group G_{13} (see Table 1). One notes that when considering the solutions on subgroups containing transformations with the operator X_{10} it is advisable to make use of the cylindrical coordinate system,

$$x,\ r = \sqrt{y^2 + z^2},\ \varphi = \operatorname{arctg}(y/z),$$
$$v_r = v \sin\varphi + w \cos\varphi,\ v_\varphi = v \cos\varphi - w \sin\varphi.$$

Subgroup 1: $X_1 + X_6$, $2X_5 - X_9$, X_{10}. The invariant solution is given by

$$u = r^{1/2}U(\lambda) + t,\ v_r = r^{1/2}V(\lambda),\ v_\varphi = r^{1/2}W(\lambda),$$
$$\rho = r^{-1}R(\lambda),\ p = P(\lambda),$$

where $\lambda = (t^2 - 2x)/r$.

By substituting these expressions into the system (1) one finds the system S/H,

$$2UU' + V(\lambda U' - U/2) + 2P'/R - 1 = 0,$$
$$2UV' + V(\lambda V' - V/2) + W^2 + (\lambda/R)P' = 0,$$
$$2UW' + V[\lambda W' - (3/2)W] = 0,\ 2(UR)' + \lambda(VR)' - VR/2 = 0,$$
$$2UP' + \lambda VP' + \gamma P[2U' - (3/2)V + \lambda V'] = 0$$

(here and in our further considerations the prime denotes differentiation).

By setting V = 0 one can find a particular solution of the system, namely,

$$U = 0,\ V = 0,\ W = [-\lambda/2]^{1/2},\ R = 2P'$$

(P(λ) is an arbitrary function λ).

TABLE 2.

X_4	X_{10}	$X_5-X_9+X_{14}$
		X_2+X_{14}
	$\alpha X_5-\alpha X_9+X_{10}$	X_{14}
	X_2+X_{10}	
		X_2+X_{14}
	X_5-X_9	$X_{10}+X_{14}$
		X_{14}
	X_7	$X_5-X_9+X_{14}$
		X_2+X_{14}
		X_{14}
		$\alpha X_2+X_4+X_{14}$
X_5-X_9	X_{10}	αX_1+X_{14}
		X_{14}
X_3+X_6	X_7	$X_5-X_9+X_{14}$
		$\alpha X_2+\beta X_3+\delta X_4+X_{14}$

The corresponding solution of the system (1) can be written as follows:

$$u = t,\ v_r = 0,\ v_\varphi = \sqrt{x - t^2/2}, \tag{2}$$
$$\rho = (2/r)P'((t^2 - 2x)/r),\ p = P((t^2 - 2x)/r),$$

where P is an arbitrary function.

Subgroup 2: X_1, $-\alpha X_9 + X_{10}$, $X_5 - \beta X_9$. The invariant solution is given by

$$u = r^\beta e^{\alpha\varphi} U(\lambda),\ v_r = r^\beta e^{\alpha\varphi} V(\lambda),\ v_\varphi = r^\beta e^{\alpha\varphi} W(\lambda),$$
$$\rho = r^{-2\beta} e^{-2\alpha\varphi} R(\lambda),\ p = P(\lambda),$$

where $\lambda = x/r$. The functions U, V, W, R, P are solutions of the following system of ordinary differential equations:

$$U'(U - \lambda V) + U(\beta V + \alpha W) + P'/R = 0,$$
$$V'(U - \lambda V) + V(\beta V + \alpha W) - W^2 - \lambda P'/R = 0,$$
$$W'(U - \lambda V) + W[(\beta + 1)V + \alpha W] = 0,$$
$$(UR)' - \lambda(VR)' + R[(1 - \beta)V - \alpha W] = 0,$$
$$P'(U - \lambda V) + \gamma P[U' - \lambda V' + (\beta + 1)V + \alpha W] = 0.$$

One notes a simplification in the case $\beta = -2$, $\alpha = 0$. Then the last two equations have the following integrals:

$$(U - \lambda V)R = C_1,\ P = C_2(U - \lambda V)^{-\gamma},$$

where C_1, C_2 are arbitrary constants.

In the case $\beta = -2$, $W = 0$ the third equation is satisfied identically, and the last two equations possess the same integrals.

Subgroup 3: X_1, $X_2 - X_9$, $-\alpha X_9 + X_{10}$. Invariant solutions are given by

$$u = e^{\alpha\varphi+x} U(\lambda),\ v_r = e^{\alpha\varphi+x} V(\lambda),\ v_\varphi = e^{\alpha\varphi+x} W(\lambda),$$
$$\rho = e^{-2(\alpha\varphi+x)} R(\lambda),\ p = P(\lambda),\ \lambda = r.$$

By inserting these expressions into the system (1), the equations

$$U'V + U[U + (\alpha/\lambda)W] = 0,$$
$$V'V + V[U + (\alpha/\lambda)W] - (1/\lambda)W^2 + P'/R = 0,$$
$$W'V + W[U + (\alpha/\lambda)W + (1/\lambda)V] = 0,$$
$$(VR)' - R[U + (\alpha/\lambda)W - (1/\lambda)V] = 0,$$
$$VP + \gamma P[U + (\alpha/\lambda)W + (1/\lambda)V + V'] = 0,$$

are obtained. Setting V = 0, the following solution of the system (1) can be obtained:

$$u = -e^{\alpha\varphi+x}(\alpha/r^{1/2})(P'/R)^{1/2},\ v_r = 0,$$
$$v_\varphi = e^{\alpha\varphi+x}(rP'/R)^{1/2},\ \rho = e^{-2(\alpha\varphi+x)}R(r),\ p = P(r), \tag{3}$$

where R and P are arbitrary functions.

Subgroup 4: X_6, $\alpha X_5 + X_9$, $\beta X_5 + X_{10}$. Invariant solution is given by

$$u = (1/t)(rU(\lambda) + x),\ v_r = (r/t)V(\lambda),\ v_\varphi = (r/t)W(\lambda),$$
$$\rho = (t^2/r^2)R(\lambda),\ p = P(\lambda),\ \lambda = r^{\alpha+1}t^{-\alpha}e^{-\beta\varphi}.$$

The system S/H is as follows:

$$\lambda U'[(\alpha+1)V - \alpha - \beta W] + UV = 0,$$
$$\lambda V'[(\alpha+1)V - \alpha - \beta W] + V(V-1) - W^2 + (\alpha+1)\lambda P'/R = 0,$$
$$\lambda W'[(\alpha+1)V - \alpha - \beta W] + W(2V-1) - \beta\lambda P'/R = 0,$$
$$3R - \alpha\lambda R' + (\alpha+1)\lambda(VR)' - \beta\lambda(WR)' = 0,$$
$$\lambda P'[(\alpha+1)V - \alpha - \beta W] + \gamma P[1 + 2V + (\alpha+1)\lambda V' - \beta\lambda W'] = 0.$$

By setting $\alpha = -1$, $\beta = 0$, $W = 0$ the exact solution can be obtained of the system (1), namely,

$$u = (r/t)C_1/(C_2+t) + x/t,\ v_r = r/(C_2+t),$$
$$v_\varphi = 0,\ \rho = C_3/tr^2,\ p = C_4/t^\gamma(t+C_2)^{2\gamma},$$

where C_1, C_2, C_3, C_4 are arbitrary constants.

Subgroup 5: X_6, X_7, $X_3 + X_9$ (contains no transformations with the operator X_{10}). The invariant solution is as follows;

$$u = [U(\lambda) + x]/t,\ v = [V(\lambda) + y - \ln t]/t,$$
$$w = W(\lambda)/t,\ \rho = t^2R(\lambda),\ p = P(\lambda),\ \lambda = z.$$

The substitution into the system (1) results in the equations

$$WU' = 0,\ WV' = 1,\ -W + WW' + P'/R = 0,$$
$$4R + (WR)' = 0,\ WP' + \gamma P(2 + W') = 0.$$

It follows from the first equation that either W = 0 or U' = 0. As regards the second equation one adopts $U = C_1$ (C_1 is a constant), $W \neq 0$. By setting $W = A\lambda$ (A is a constant) one obtains the following partial solution of the system (1):

$$u = (x + C_1)/t,\ v = y/t + (\ln z)/At - (\ln t)/t,$$
$$w = Az/t,\ \rho = C_2t^2z^{-2(2+\gamma)/(2-\gamma)},\ p = C_3z^{3\gamma/(\gamma-2)},$$

where

$$A = (4-2\gamma)/(\gamma+1);\ C_3 = 2C_2(1-\gamma)(2-\gamma)^2/\gamma(1+\gamma)^2;$$

C_1, C_2 being arbitrary constants.

It is noted that the system S/H on the subgroup $\langle X_6, X_7, X_8\rangle$ can be integrated completely. The solution of the system (1) is then given by

$$u = (x + A_1)/t,\ v = (y + A_2)/t,\ w = (z + A_3)/t,$$
$$\rho = A_4t^{-3},\ p = A_5t^{-3\gamma} \tag{4}$$

(A_1 ... A_5 being arbitrary constants).

By carrying out the translation in the space coordinates the above solution can be reduced to a simpler form,

$$u = x/t,\ v = y/t,\ w = z/t,\ \rho = C_1t^{-3},\ p = C_2t^{-3\gamma},$$

where C_1, C_2 are arbitrary constants. Of course, the obtained solution is self-consistent.

We now go over to the case of the adiabatic index $\gamma = 5/3$. The solutions will be found on some of the subgroups shown in Table 2.

Subgroup 6: X_6, $\alpha X_5 - \alpha X_9 + X_{10}$, X_{14}. The invariant solution is given by

$$u = [rU(\lambda)/t + x]/t,\ v_r = [rV(\lambda)/t + r]/t,$$
$$v_\varphi = rW(\lambda)/t^2,\ \rho = R(\lambda)/r^2 t,\ p = P(\lambda)/t^5,$$
$$\lambda = \ln(r/t) - \alpha\varphi.$$

The system S/H can be written for this solution as

$$U'(V - \alpha W) + UV = 0,\ V'(V + \alpha W) + V^2 + W^2 + P'/R = 0,$$
$$W'(V - \alpha W) + 2VW - \alpha P'/R = 0,\ (VR)' = \alpha(WR)',$$
$$P'(V - 2W) + (5/3)P(2V + V' - \alpha W') = 0.$$

By setting $\alpha = 0$, $V = 0$ one finds that $W^2 = P'/R$; the latter yields the following particular solution of the system (1):

$$u = U(r/t)/t + x/t,\ v_r = r/t,$$
$$v_\varphi = (rP'/tR)^{1/2} r/t^2,\ \rho = R(r/t)/r^2 t,\ p = P(r/t)t^{-5}, \tag{5}$$

where U, R, P are arbitrary functions.

Subgroup 7: $X_5 - X_9$, X_{10}, $X_1 + X_{14}$. The invariant solution is

$$u = x[U(\lambda) + t]/(1 + t^2),\ v_r = r[V(\lambda) + t]/(1 + t^2),$$
$$v_\varphi = xW(\lambda)/(1 + t^2),\ \rho = x^{-2}(1 + t^2)^{-1/2}R(\lambda),$$
$$p = P(\lambda)(1 + t^2)^{-5/2},\ \lambda = x/r,$$

and the system S/H is as follows:

$$\lambda U'(U - V) + U^2 + (\lambda/R)P' + 1 = 0,$$
$$\lambda V'(U - V) + V^2 + \lambda^2 W^2 - \lambda^3 P'/R + 1 = 0,$$
$$\lambda W'(U - V) + W(U + V) = 0,$$
$$(UR)' - (VR)' + R(2V - U)/\lambda = 0,$$
$$\lambda P'(U - V) + (5/3)P(U + \lambda U' + 2V - \lambda V') = 0.$$

The following set of functions $U = V = 0$, $W^2 = 1 + 1/\lambda^2$, $R = -\lambda P'$ is one of the particular solutions for this system. In this case the solution of the system (1) is given by

$$u = xt/(1 + t^2),\ v_r = rt/(1 + t^2),$$
$$v_\varphi = \pm(\sqrt{r^2 + x^2})/(1 + t^2),\ \rho = -P'(x/r)/xr\sqrt{1 + t^2}, \tag{6}$$
$$p = (1 + t^2)^{-5/2}P(x/r)$$

(P is an arbitrary function).

Let us consider the case in which the system S/H can be completely integrated. This is possible for the subgroup $\langle X_6, X_7, X_2 + X_{14}\rangle$. The invariant solution on this subgroup is

$$u = [U(\lambda) + x + 1/t]/t,\ v = [V(\lambda) + y]/t,$$
$$w = [W(\lambda) + z]/t,\ \rho = R(\lambda)t^{-3},\ p = P(\lambda)t^{-5},\ \lambda = z/t.$$

Having solved the system S/H and by using the space-coordinates transfer as well as by eliminating some integration, constants one obtains the invariant solution,

$$u = C_1 z/t^2 + x/t,\ v = y/t,\ w = z/t,\ \rho = C_2 t^{-3},\ p = C_3 t^{-5},$$

where C_1, C_2, C_3 are arbitrary constants.

The system S/H is also completely integrated for the subgroup $\langle X_6, X_7, X_{14}\rangle$. The solution of the system (1) is then as follows:

$$u = (C_1 + x)/t,\ v = (C_2 + y)/t,\ w = (C_3 + z)/t,\ \rho = C_4 t^{-3},\ p = C_5 t^{-5}$$

[this solution is obtained by using (4)].

We analyze in more detail those of the obtained solutions of the system (1) in which arbitrary functions appear. For these solutions the contact characteristics are found which are specified by an equation of the form

$$x = f_1(r, \varphi) + l_1(t),\ r = f_2(\varphi) + l_2(t),\ \varphi = l_3(t). \tag{7}$$

It is the contact characteristics that are analyzed since it is convenient to join the obtained solutions to other solutions or to replace them by impenetrable surfaces. The following surfaces are contact characteristics for the solution (2): $x = f(r) + t^2/2$, $r = C$ (f is an arbitrary function, C is an arbitrary constant). As regards coordinate system in motion along the x axis, with constant acceleration equal to 1, the gas motion governed by this solution will be stationary. It should be mentioned that the surfaces $x = C_1 r + t^2/2$, which are contact characteristics, are isobaric surfaces at the same time.

The solution (3) is now considered; it governs the stationary gas motion. In this case the contact characteristics are given by the equations $x = -\alpha\varphi + f(r)$, $r = C$ (f is an arbitrary function, C is an arbitrary constant). The pressure on the cylindrical surfaces $r = C$ is in this case constant. Employing surfaces specified by the equations $x = \alpha\varphi$, $r = C_1$, $r = C_2$ one can obtain a body in the form of a screw-like tube with rectangular cross section. The obtained solution gives the distribution of the flow parameters of the gas for the flow inside such a tube.

The contact characteristics for the solution in (5) are specified by the formula $r = Ct$ (C is an arbitrary constant). If in the solution one sets $U = C_1$ then the expression can be written down for another family of characteristics: $x = C_2 r + C_3 t - C_1$ (C_1, C_2, C_3 are arbitrary constants). If one selects the function P such that $P(A) = 0$ (A is a constant) the solution under consideration can be regarded as the solution of the problem on the scattering into a cavity of a cylindrical gas volume with a given initial velocity field and with an initial distribution of density and pressure. The boundary of this volume is in motion according to the rule $r = At$.

For the solution (6) the surfaces given by the equation $x = C_1 r + C_2\sqrt{1 + t^2}$ are the contact characteristics of the type as in (7) (C_1, C_2 are arbitrary constants).

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